Cohen-Macaulay normal Rees algebras of integrally closed ideals

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Introduction

Throughout this talk, unless otherwise specified, let

Setup 1

- $(R, \mathfrak{m})$ *a RLR with* $d = \dim R$
- I *an integrally closed* $\mathfrak{m}$*-primary ideal of* R.

Question 2

When does the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ become a CM normal domain?

- Question 2 is always true when $d \leq 1$.
- As R is a normal domain, we have $\overline{\mathcal{R}(I)}^{\mathrm{Q}(\mathcal{R}(I))} = \bigoplus_{n \geq 0} \overline{I^n}$.
- $\mathcal{R}(I)$ is normal $\iff$ I is normal, i.e., $\overline{I^n} = I^n$ for $\forall\, n \geq 1$.

Preceding results

- If $d = 2$, then $\mathcal{R}(I)$ is normal ([Zariski, 1938], [Zariski-Samuel, 1960]).
- If $d = 2$, then $\mathcal{R}(I)$ is CM ([Lipman-Teissier, 1981]).

- If R is a two-dimensional rational singularity, then $\mathcal{R}(I)$ is a CM normal domain, provided $|R/\mathfrak{m}| = \infty$ ([Lipman, 1969]).

Recall that a normal local ring $(R, \mathfrak{m})$ is a rational singularity if $\exists$ resolution of singularity $f : X \to \operatorname{Spec} R$ s.t. $\mathrm{H}^i(X, \mathcal{O}_X) = (0)$ for $\forall\, i > 0$.

Preceding results ([Cutkosky, 1990])

Let $(R, \mathfrak{m})$ be an excellent normal local domain with $\dim R = 2$. Suppose that $R/\mathfrak{m}$ is algebraically closed. Then TFAE.

(1) R is a rational singularity.

(2) If I and J are integrally closed $\mathfrak{m}$-primary ideals of R, then IJ is integrally closed.

(3) If I is an integrally closed $\mathfrak{m}$-primary ideal of R, then I^2 is integrally closed.

By [Cutkosky, 1990], the ring

$$R = \mathbb{Q}[[X, Y, Z]]/(X^3 + 3Y^3 + 9Z^3)$$

is a non-rational normal local domain with $\dim R = 2$, and is a simple elliptic singularity. Besides, for all integrally closed $\mathfrak{m}$-primary ideals I and J of R, IJ is integrally closed.

When $d \geq 3$, we have the following examples.

Example 3

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k. Consider

$$Q = (X^7, Y^3, Z^2) \quad \text{and} \quad I = \overline{Q} = (X^7, Y^3, Z^2, X^5Y, X^4Z, X^3Y^2, X^2YZ, Y^2Z).$$

Then $\overline{I} = I$, $\overline{I^2} \neq I^2$, and $I^2 = QI$. Hence $\mathcal{R}(I)$ is CM, but not normal.

The ideal I as in Example 3 is the contraction of $k \cdot t^{42} + t^{44}\overline{A}$ where $A = k[[t^6, t^{14}, t^{21}]]$.

Example 4 ([Huckaba-Huneke, 1999])

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k. Suppose $\operatorname{ch} k \neq 3$. Consider

$$I = (X^4, X(Y^3 + Z^3), Y(Y^3 + Z^3), Z(Y^3 + Z^3)) + \mathfrak{m}^5$$

where $\mathfrak{m} = (X, Y, Z)$. Then I is normal and $\operatorname{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is not CM. Hence, $\mathcal{R}(I)$ is normal, but not CM.

Let $v(-)$ denote the embedding dimension of a ring.

Preceding results

- By [Goto, 1987], we have

  (1) $\mu_R(I) = d \implies \mathcal{R}(I)$ is a CM normal domain

  (2) $\mu_R(I) = d \iff v(R/I) \leq 1$.

- By [Ciupercă, 2006, 2011], we have

  (1) $\mu_R(I) = d + 1 \implies \mathcal{R}(I)$ is a CM normal domain

  (2) $\mu_R(I) = d + 1 \implies v(R/I) \leq 2$.

With this observation, one of the main results of this talk is stated as follows.

Theorem A

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $\sqrt{I} = \mathfrak{m}$ s.t. $\overline{I} = I$. Then

(1) $v(R/I) \leq 2 \implies \mathcal{R}(I)$ *is a CM normal domain*

(2) $\mu_R(I) \leq d + 2 \implies v(R/I) \leq 2$.

In particular, if $\mu_R(I) \leq d + 2$, then $\mathcal{R}(I)$ is a CM normal domain.

Preliminaries

In this section, let R be a Noetherian ring and I an ideal of R.

- An element $x \in R$ is called integral over I, if
$$x^n + c_1x^{n-1} + \cdots + c_n = 0 \quad \text{for } \exists\, n \geq 1,\ \exists\, c_i \in I^i\ (1 \leq i \leq n).$$
- We set
$$I \ \subseteq\ \overline{I} = \{x \in R \mid x \text{ is integral over } I\} \ \subseteq\ R$$
which forms an ideal of R, and call it the integral closure of I.
- We say that
  - I is integrally closed, if $\overline{I} = I$
  - I is normal, if $\overline{I^n} = I^n$ for $\forall\, n \geq 1$.

We define
$$\mathcal{R}(I) = R[It] = \sum_{n \geq 0} I^n t^n \subseteq R[t], \quad \mathcal{R}(I) \cong \bigoplus_{n \geq 0} I^n$$
and call it the Rees algebra of I.

- The canonical morphism $f : \operatorname{Proj} \mathcal{R}(I) \to \operatorname{Spec} R$ is the blow-up of $\operatorname{Spec} R$ along the subscheme $V(I)$ defined by I.
- Rees algebras also arise as the bihomogeneous coordinate rings of graphs of rational maps between projective spaces.

Note that

$$\overline{\mathcal{R}(I)}^{R[t]} = \sum_{n\geq 0} \overline{I^n} t^n \cong \bigoplus_{n\geq 0} \overline{I^n} \quad \text{and} \quad \overline{\mathcal{R}(I)}^{Q(\mathcal{R}(I))} = \sum_{n\geq 0} \overline{I^n \overline{R}} t^n \cong \bigoplus_{n\geq 0} \overline{I^n \overline{R}}.$$

Hence, $\mathcal{R}(I)$ is normal $\iff$ I is normal, provided R is a normal domain.

The associated graded ring of I

$$\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1} \cong R/I \otimes_R \mathcal{R}(I)$$

plays a key role in the study of $\mathcal{R}(I)$.

Theorem 5 ([Goto-Shimoda, 1979])

Let $(R, \mathfrak{m})$ be a CM local ring with $\dim R \geq 1$ and $\sqrt{I} = \mathfrak{m}$. Then

$$\mathcal{R}(I) \text{ is CM} \iff \mathrm{gr}_I(R) \text{ is CM and } \mathrm{a}(\mathrm{gr}_I(R)) < 0.$$

- Theorem 5 holds for ideals I with $\mathrm{ht}_R\, I > 0$ ([Trung-Ikeda, 1989]).
- Theorem 5 holds for filtrations of ideals/modules ([Goto-Nishida, 1994], [Viet, 1993], [T-Phuong-Dung-An, 2017]).
- When R is a RLR (or more generally pseudo-rational local ring) and $I \neq R$, we have

$$\mathcal{R}(I) \text{ is CM} \iff \mathrm{gr}_I(R) \text{ is CM} \qquad \text{([Lipman, 1994]).}$$

We prove Theorem A by induction on dim R.

Example 6

Let $(R, \mathfrak{m})$ be a RLR with $\dim R = 2$ and $\mathfrak{m} = (x, y)$. We consider $I = \mathfrak{m}^2 = (x^2, xy, y^2)$. Then $\overline{I} = I$, but $\overline{I/(x^2)} \neq I/(x^2)$, because $\overline{x} \notin I/(x^2)$.

Theorem 7 ([Hong-Ulrich, 2014] The preprint was posted on arXiv in 2006.)

Let $(R, \mathfrak{m})$ be a Noetherian, equi-dimensional, universally catenary local ring s.t. $R/\sqrt{(0)}$ is analytically unramified. Let $I = (a_1, a_2, \ldots, a_n)$ be an ideal of R with $\mathrm{ht}_R I \geq 2$. Set

$$R'' = R[Z_1, Z_2, \ldots, Z_n]_{mR[Z_1, Z_2, \ldots, Z_n]}, \quad I'' = IR'', \quad \text{and} \quad x = \sum_{i=1}^{n} Z_i a_i.$$

Then

$$\overline{I''/(x)} = \overline{I''}/(x).$$

Theorem 8 ([Ciupercă, 2011])

Let $(R, \mathfrak{m})$ be a RLR and I an ideal of R s.t. $I \not\subseteq \mathfrak{m}^2$. For each $\forall x \in I \setminus \mathfrak{m}^2$, we have

$$\overline{I/(x)} = \overline{I}/(x).$$

Proof of Theorem A

Theorem A

Let $(R, \mathfrak{m})$ *be a RLR with* $d = \dim R$ *and* $\sqrt{I} = \mathfrak{m}$ *s.t.* $\overline{I} = I$*. Then*

(1) $v(R/I) \leq 2 \implies \mathcal{R}(I)$ *is a CM normal domain*

(2) $\mu_R(I) \leq d + 2 \implies v(R/I) \leq 2$.

In particular, if $\mu_R(I) \leq d + 2$*, then* $\mathcal{R}(I)$ *is a CM normal domain.*

(Proof) (1) We may assume $d \geq 3$ and the assertion holds for $d - 1$. Choose a regular subsystem $a_1, a_2, \dots, a_{d-2}$ of parameters of R s.t. $I = (a_1, a_2, \dots, a_{d-2}, a_{d-1}, \dots, a_n)$. Let

$$R'' = R[Z_1, Z_2, \dots, Z_n]_{\mathfrak{m}R[Z_1, Z_2, \dots, Z_n]}, \quad I'' = IR'', \quad \mathfrak{m}'' = \mathfrak{m}R'', \quad \text{and} \quad x = \sum_{i=1}^n Z_i a_i.$$

Since $x \notin (\mathfrak{m}'')^2$, we see that $R''/(x)$ is a RLR with $\dim R''/(x) = d - 1$. Besides

$$\sqrt{I''/(x)} = \mathfrak{m}''/(x) \quad \text{and} \quad \overline{I''/(x)} = \overline{I''}/(x) = I''/(x).$$

Moreover, because $v(R/I) \leq 2$ and $R''/(x)$ is regular, we obtain

$$v(R''/(x)/I''/(x)) \leq 2.$$

The induction argument shows $\mathcal{R}(I''/(x))$ is CM normal. In particular, $\mathrm{gr}_{I''/(x)}(R''/(x))$ is CM. Let $\mathcal{F} = \left\{\overline{(I'')^n}\right\}_{n\in\mathbb{Z}}$ and consider

$$G(\mathcal{F}) = \bigoplus_{n\geq 0} \overline{(I'')^n}/\overline{(I'')^{n+1}} = \mathcal{R}'(\mathcal{F})/t^{-1}\mathcal{R}'(\mathcal{F}), \text{ where } \mathcal{R}'(\mathcal{F}) = \sum_{n\in\mathbb{Z}} \overline{(I'')^n}t^n \subseteq R''[t,t^{-1}].$$

Then $xt \in \mathcal{R}'(\mathcal{F})$ is a NZD on $G(\mathcal{F})$ ([Hong-Ulrich, 2014]). As $I''/(x)$ is normal, we get

$$\overline{(I'')^n} \subseteq (I'')^n + (x)$$

for $\forall\, n \geq 1$. Thus

$$\overline{(I'')^n} = \left[(I'')^n + (x)\right] \cap \overline{(I'')^n} = (I'')^n + \left[(x) \cap \overline{(I'')^n}\right] = (I'')^n + x \cdot \overline{(I'')^{n-1}}$$

because $(x) \cap \overline{(I'')^n} \subseteq x \cdot \overline{(I'')^{n-1}}$. The induction on $n \geq 1$ shows

$$\overline{(I'')^n} = (I'')^n \qquad (\forall\, n \geq 1)$$

so that $\mathcal{R}(I'')$ is normal. Therefore, $G(\mathcal{F}) = \mathrm{gr}_{I''}(R'')$ and is CM, because

$$\mathrm{gr}_{I''}(R'')/(xt)\,\mathrm{gr}_{I''}(R'') \cong \mathrm{gr}_{I''/(x)}(R''/(x)).$$

Thus $\mathcal{R}(I'')$ is CM. Since $\mathcal{R}(I) \to R'' \otimes_R \mathcal{R}(I) \cong \mathcal{R}(I'')$ is faithfully flat, we conclude that $\mathcal{R}(I)$ is a CM normal domain.

(2) May assume $d \geq 3$, $|R/\mathfrak{m}| = \infty$. As $\sqrt{I} = \mathfrak{m}$ and $\overline{I} = I$, the ideal I is $\mathfrak{m}$-full. Then

$$I \not\subseteq \mathfrak{m}^2.$$

Indeed, if $I \subseteq \mathfrak{m}^2$, then

$$d + 2 \geq \mu_R(I) \geq \mu_R(\mathfrak{m}^2) = \binom{d+1}{2} = \frac{d(d+1)}{2}.$$

This makes a contradiction because $d \geq 3$. Hence $I \not\subseteq \mathfrak{m}^2$. So, we may assume $d \geq 4$ and the assertion holds for $d-1$. Choose $x \in I$ s.t. $x \notin \mathfrak{m}^2$. Then

$$\overline{I/(x)} = \overline{I}/(x) = I/(x).$$

Moreover, $R/(x)$ is a RLR with $\dim R/(x) = d-1$, $\sqrt{I/(x)} = \mathfrak{m}/(x)$, and

$$\mu_{R/(x)}(I/(x)) = \mu_R(I) - 1 \leq (d+2) - 1 = (d-1) + 2.$$

By induction hypothesis, we have $v(R/(x)/I/(x)) \leq 2$. Since $R/(x)$ is regular, we get

$$v(R/I) \leq 2$$

as desired. □

Example 9

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k.

- Let $I = \overline{(X^3, Y^3, Z)} = (X^3, X^2Y, XY^2, Y^3, Z)$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $\mu_R(I) = 5 = d + 2$. Hence, $\mathcal{R}(I)$ is a CM normal domain.
- Let $I = \overline{(X^4, Y^4, Z)} = (X^4, X^3Y, X^2Y^2, XY^3, Y^4, Z)$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $\mu_R(I) = 6 > d + 2$, but $v(R/I) = 2$. Hence, $\mathcal{R}(I)$ is a CM normal domain.
- Let $I = (f) + \mathfrak{m}^n$ for $\forall\, f \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\forall\, n \geq 1$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $v(R/I) \leq 2$. Hence, $\mathcal{R}(I)$ is a CM normal domain.
- Let $I = \overline{(X^2, Y^2, Z^4)} = (X^2, XY, Y^2, Z^4, XZ^2, YZ^2) \subseteq \mathfrak{m}^2$. Then $\overline{I} = I$, $\sqrt{I} = \mathfrak{m}$, and $v(R/I) = 3$. Since $\overline{I^2} = I^2$, the ideal I is normal. By setting $Q = (X^2, Y^2, Z^4)$, we have $I^2 = QI$. Hence, $\mathcal{R}(I)$ is a CM normal domain.

Theorem 10 ([Reid-Roberts-Vitulli, 2003])

Let $R = k[X_1, X_2, \ldots, X_d]$ be the polynomial ring over a field k and I a monomial ideal s.t. $\overline{I^i} = I^i$ for $1 \leq \forall\, i \leq d - 1$. Then I is normal.

Rees algebras of integrally closed modules

Theorem A

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $\sqrt{I} = \mathfrak{m}$ s.t. $\overline{I} = I$. Then

(1) $v(R/I) \le 2 \implies \mathcal{R}(I)$ *is a CM normal domain*

(2) $\mu_R(I) \le d + 2 \implies v(R/I) \le 2$.

In particular, if $\mu_R(I) \le d + 2$, then $\mathcal{R}(I)$ is a CM normal domain.

Theorem A can be extended to the Rees algebras of integrally closed modules.

Theorem B

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $F = R^{\oplus e}$ $(e > 0)$. Let E be an R-submodule of F s.t. $\ell_R(F/E) < \infty$ and $\overline{E} = E$. Then

(1) $\mu_R([E + \mathfrak{m}F]/E) \le 2 \implies \mathcal{R}(E)$ *is a CM normal domain*

(2) $\mu_R(E) \le d + e + 1 \implies \mu_R([E + \mathfrak{m}F]/E) \le 2$.

In particular, if $\mu_R(E) \le d + e + 1$, then $\mathcal{R}(E)$ is a CM normal domain.

Let R be a Noetherian ring and $E \subseteq F = R^{\oplus e}$ $(e > 0)$ an R-submodule of F. Let

$$\mathrm{Sym}(i) : \mathrm{Sym}_R(E) \to \mathrm{Sym}_R(F) = R[t_1, t_2, \ldots, t_e] =: S$$

be the induced homomorphism of $E \subseteq F$. Then the Rees algebra of E is defined by

$$\mathcal{R}(E) = \mathrm{Im}\,\mathrm{Sym}(i) \subseteq S = R[t_1, t_2, \ldots, t_e] = \bigoplus_{n \geq 0} E^n.$$

- $\mathcal{R}(E)$ is a standard graded algebra over R with $E^1 = E$.
- If $E = I$, then $\mathcal{R}(E) = \mathcal{R}(I)$.
- If $E = I_1 \oplus I_2 \oplus \cdots \oplus I_r$, then $\mathcal{R}(E) = \mathcal{R}(I_1, I_2, \ldots, I_r) = R[I_1 t_1, I_2 t_2, \ldots, I_r t_r]$.
- When $E = I_1 \oplus I_2 \oplus \cdots \oplus I_r$, $\mathcal{R}(E)$ arises in successive blow-up of $\mathrm{Spec}\, R$ at the subschemes defined by $I_1, I_2, \ldots, I_r$.
- Gaffney required $\mathcal{R}(E)$ for applications to equisingularity theory.

For $\forall\, n \geq 0$, we define

$$\overline{E^n} = \left[\overline{\mathcal{R}(E)}^S\right]_n = \left[\overline{(ES)^n}\right]_n \subseteq S_n = F^n.$$

In particular

$$\overline{E} = \left[\overline{ES}\right]_1 = \{x \in F \mid x^n + c_1 x^{n-1} + \cdots + c_n = 0 \text{ for } \exists\, n \geq 1, \exists\, c_i \in E^i\ (1 \leq i \leq n)\}.$$

We say that

- E is integrally closed, if $\overline{E} = E$
- E is normal, if $\overline{E^n} = E^n$ for $\forall\, n \geq 1$.

When R is a normal domain, we have

$$\overline{\mathcal{R}(E)}^S = \overline{\mathcal{R}(E)}^{\mathrm{Q}(\mathcal{R}(E))}.$$

Hence, $\mathcal{R}(E)$ is normal $\iff$ E is normal.

For an arbitrary module, one lacks the remarkable interaction that exists for an ideal $I \subseteq R$ between $\mathcal{R}(I)$ and $\mathrm{gr}_I(R)$.

Generic Bourbaki ideals ([Simis-Ulrich-Vasconcelos, 2003])

Suppose that $(R, \mathfrak{m})$ is a local ring and $\exists$ rank $E = e > 0$. Assume that $E_{\mathfrak{p}}$ is free for $\forall\, \mathfrak{p} \in \operatorname{Spec} R$ with depth $R_{\mathfrak{p}} \leq 1$. Set

- $E = Ra_1 + Ra_2 + \cdots + Ra_n$ $(a_i \in E)$
- $Z = \{Z_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq e-1\}$
- $R'' = R[Z]_{\mathfrak{m}R[Z]}$, $E'' = E \otimes_R R''$, $x_j = \sum_{i=1}^n Z_{ij} a_i$ $(1 \leq j < e)$, $G'' = \sum_{j=1}^{e-1} R'' x_j$.

Then

$$G'' \cong (R'')^{\oplus(e-1)}, \quad E''/G'' \cong \exists\, I \subseteq R'' \text{ ideal of } R'' \text{ s.t. } \mathrm{grade}_{R''} I > 0$$

The ideal I is called the generic Bourbaki ideal of E.

With this notation, we have

- $\mathcal{R}(E)$ is CM $\iff$ $\mathcal{R}(I)$ is CM
- $\mathcal{R}(E)$ is normal $\Longleftarrow$ $\mathcal{R}(I)$ is normal, and the converse holds if

$$\operatorname{depth} R_{\mathfrak{p}} \otimes_R \mathcal{R}(E) \geq e+1 \text{ for } (0) \neq \forall\, \mathfrak{p} \in \operatorname{Spec} R.$$

If any of the conditions of above hold, then

$$\mathcal{R}(E'')/(G'') \cong \mathcal{R}(I) \quad \text{and} \quad x_1, x_2, \ldots, x_{e-1} \text{ forms a regular sequence on } \mathcal{R}(E'').$$

Theorem B

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $F = R^{\oplus e}$ $(e > 0)$. Let E be an R-submodule of F s.t. $\ell_R(F/E) < \infty$ and $\overline{E} = E$. Then

(1) $\mu_R([E + \mathfrak{m}F]/E) \leq 2 \implies \mathcal{R}(E)$ *is a CM normal domain*

(2) $\mu_R(E) \leq d + e + 1 \implies \mu_R([E + \mathfrak{m}F]/E) \leq 2.$

In particular, if $\mu_R(E) \leq d + e + 1$, then $\mathcal{R}(E)$ is a CM normal domain.

(Proof) We may assume $E \neq F$ and $d \geq 2$. Let

- $E = Ra_1 + Ra_2 + \cdots + Ra_n$ $(a_i \in E)$
- $Z = \{Z_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq e-1\}$
- $R'' = R[Z]_{\mathfrak{m}R[Z]}$, $E'' = E \otimes_R R''$, $x_j = \sum_{i=1}^n Z_{ij} a_i$ $(1 \leq j < e)$, $G'' = \sum_{j=1}^{e-1} R'' x_j$.

Then

$$G'' \cong (R'')^{\oplus(e-1)}, \quad E''/G'' \cong \exists\, I \subseteq R'' \text{ ideal of } R'' \text{ s.t. } \mathrm{grade}_{R''} I > 0.$$

By setting $F'' = F \otimes_R R''$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G'' & \longrightarrow & E'' & \longrightarrow & E''/G'' \cong I & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & G'' & \longrightarrow & F'' & \longrightarrow & F''/G'' \cong R'' & \longrightarrow & 0
\end{array}$$

so that $(0) \neq F''/E'' \cong R''/I$. This shows $\sqrt{I} = \mathfrak{m}''$. As $\mathrm{grade}''_R I = d \geq 2$, the sequence

$$0 \to G'' \to E'' \to I \to 0$$

splits. Hence $E'' = G'' \oplus I$. Therefore

$$[E'' + \mathfrak{m}F'']/E'' = \frac{G'' \oplus \mathfrak{m}''}{G'' \oplus I} \cong \mathfrak{m}''/I.$$

This implies $v(R''/I) = \mu_{R''}(\mathfrak{m}''/I) = \mu_R([E + \mathfrak{m}F]/E) \leq 2$. Besides, we can check $\overline{I} = I$ by using $\overline{E} = E$. Therefore, by Theorem A, $\mathcal{R}(I)$ is a CM normal domain. Hence, by [Simis-Ulrich-Vasconcelos, 2003], $\mathcal{R}(E)$ is a CM normal domain. $\square$

Recall that E is a parameter module in F if $\ell_R(F/E) < \infty$ and $\mu_R(E) = d + e - 1$.

Corollary 11 ([Simis-Ulrich-Vasconcelos, 2003], [Brennan-Vasconcelos, 2004])

Let $(R, \mathfrak{m})$ be a RLR with $d = \dim R$ and $F = R^{\oplus e}$ $(e > 0)$. Let E be a parameter module in F. If $\overline{E} = E$, then $\mathcal{R}(E)$ is a CM normal domain.

Example 12

Let $R = k[[X, Y, Z]]$ be the formal power series ring over a field k. Let $I = (f) + \mathfrak{m}^n$ for $\forall f \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $\forall n \geq 1$. Set

$$E = \begin{matrix} \mathfrak{m}^{\oplus(e-1)} \\ \oplus \\ I \end{matrix} \subseteq \mathfrak{m}F = \begin{matrix} \mathfrak{m}^{\oplus(e-1)} \\ \oplus \\ \mathfrak{m} \end{matrix} \subseteq F = R^{\oplus e}.$$

Then $\overline{E} = E$, $\ell_R(F/E) < \infty$, and $\mu_R([E + \mathfrak{m}F]/E) \leq 2$. Hence, $\mathcal{R}(E)$ is a CM normal domain.

Thank you for your attention.